A mixed control problem of the management of natural resources.

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Abstract

In this paper we study the optimal solutions of a model of natural resource management which allows for both impulse and continuous harvesting policies. This type of model is known in the literature as mixed optimal control problem. In the resource management context, each type of control represents a different harvesting technology, which has a different cost. In particular we want to know when the following conjecture made by Clark [2005] is an optimal solution to this mixed optimal control problem: if the harvesting capacity is unlimited, it is optimal to jump immediately to the steady state of the continuous time problem and then to stay there. We show that under a particular relationship between the continuous and the impulse profit function, the conjecture made by Clark is true. In other cases, however, it is either better to use only continuous control variables or to jump to resource levels which are smaller than the steady state and then let the resource grow back to the steady state. These results emphasize the importance of the cost functions in the modelling of natural resource management.

Keywords: impulse and continuous control, impulse and continuous cost functions, natural resource management

1. Introduction

In this paper, we study an optimal control model which allows both, continuous control variables and unlimited or impulse control variables. This allows us to analyse the conjecture made by Clark: if the harvesting capacity becomes unlimited, it is optimal to jump immediately to the steady state and then to stay there.

Our model is a simple extension of the usual bioeconomic models described by Clark [2005] and is related to the fundamental ideas of turnpike theorems (see McKenzie [1976] for an overview).[1] In the optimal control literature, it is well known that the optimal solution to continuous time singular optimal control problems with one state variable consists in the Most Rapid Approach Path (MRAP) to the steady state (turnpike property). If the initial state of the system is greater (smaller) than the steady state, the solution consists in applying the upper (lower) bound of the controls until the steady state and then to stay in the steady state. However, this solution supposes that the set of admissible controls is bounded. In the case where the set of admissible controls is not bounded, there does not exist a solution to the problem.

In some cases, it can be interesting to assume unconstrained control capacities. Consider, for example, the case where the state variable represents a capital stock and the control variable, the payments withdrawn. The maximum possible amount of payments depends on the agreement between the bank and the investor and one can imagine that the whole capital stock can be withdrawn at once. Next, consider the case where the state variable represents a natural resource and the control variable the harvest. The maximum possible amount of harvest depends on the harvest technology and one should consider the fact that technological progress has shifted the relationship between stocks and harvest capacity. Indeed, nowadays, large combines

[1] As pointed by one anonymous referee, the turnpike property was first discussed by von Neumann and Ramsey but the term is often tracked back to Samuelson [1949]. See also Samuelson [1965], Dorfman et al. [1958], Sethi [1977] and Hartl and Feichtinger [1987].

are able to harvest 60 tons of grain per hour, large timber harvesters can clear cut several hectares a day and big trawlers are able to catch over 250 tons of fish a day. If the targeted stock is a particular hectare of forest or a particular surface school of fish, one might imagine the whole stock can be withdrawn at once.

Mathematically, considering unbounded controls "simply means that discontinuous jumps in the state variable, [...] are feasible" [Clark, 2005, p.58]. But as reminded above this implies that the continuous control problem considered in the first place does not have any solution. Clark describes the type of solution expected: "Under such conditions the most rapid approach solution obviously utilizes such discontinuous jump to transfer [the stock] instantly to the singular path. The control that effects such a discontinuous jump is referred to as an impulse control" [Clark, 2005, p.58]. Hence, he suggests to widen the set of admissible controls to also allow for impulse controls. With impulse controls, it is possible to remove part of the stock of the population instantaneously.

In this paper, we study the optimal solutions of a model which allows for both impulse control and continuous control variables and which is called below a mixed control problem. Our goal is to find a control that maximizes the gains in this mixed control problem. In particular we want to know whether the solution described by Clark is optimal: to jump to the steady state in the first place and then to stay there. We show that one solution to the mixed control problem is indeed the solution described by Clark, but this solution needs particular conditions to be made on the benefit or cost function. To be specific, we assume (like Clark) that unit harvest costst are c/x (with c constant), but unlike Clark, we allow for different values of c for continuous and impulse harvesting respectively. We show that Clark's claim is correct if these values are the same, i.e. if impulse gains are the integral of continuous gains, but not otherwise. In the other cases, it is either better to use only continuous control variables or to jump to resource levels which are smaller than the steady state and then let the resource grow back to the steady state. It seems indeed quite intuitive that the optimum would be to use the cheaper harvesting technique exclusively. It is however necessary to distinguish the transitory part and the long-term rate of gain, for two reasons: from a technical point of view, to ensure existance of the solution, and from a practical point of view, in all cases where the transition phase is long enough to have economic importance.

From a mathematical point of view the mixed control problem solves some kind of "compactness" of the respective controls set. On the one hand we need to introduce the impulse control because it is our mathematical tool to complete the space of controls when the continuous control is unbounded and the introduction of impulsions allows to have a solution. On the other hand, (see Erdlenbruch et al. [2013]), pure impulse control problems with integral gains do not have a solution and the introduction of continuous controls allows to have one.

The motivation of our paper is to show that one must to be careful when modelling resource management with different harvesting technologies. When the representation of the problem has changed, for example because the harvest capacity is considered to be unbounded, it is necessary to adapt not only the control variable but also the other parameters of the model, in particular the cost and benefit functions.

In our simple model the characteristics of harvesting are modelled using two different kinds of technologies, represented by different kinds of controls. This is the main extension with respect to existing models introduced by Clark. One of the controls (the continuous one) is a mathematical representation of "harvesting a bit at all times", the other one (the impulse control) is the representation of "harvesting a lot at some distant times".

The cost of these harvesting techniques can of course be different. It is thus interesting to ask whether there is one technology that is always better than the other. Or whether one should be used in the transition phase and the other in the long-term. We show that the mixed strategy: "harvesting first a lot" and "harvesting a bit at all times in the long-run" is indeed an optimal strategy, under certain conditions on the gain functions, i.e. when particular harvesting technologies are in place.

The rest of the paper is organized as follows: In Section 2, we review the solutions of the pure continuous and pure impulse control problems described in the literature and which we need to motivate our research question. In Section 3, we present the mixed control and we show that a solution can be characterized by a Hamilton-Jacobi-Bellman equation. In Section 4, we introduce particular functional forms for the dynamics and the profit functions and characterize the steady state of the continuous control problem as a potential long-term equilibrium. In Section 5, using the Hamilton-Jacobi -Bellman equation, we solve the mixed control problem under different assumptions on the cost/benefit function and show under which conditions Clark's policy is an optimal solution to our mixed control problem. We also show the non optimality of this policy in other cases. In Section 6, we make some concluding remarks.

2. Continuous and impulse control problems in the literature

In this section, first, we are going to present the usual harvesting model in continuous time in order to introduce Clark's conjecture, the need of introducing impulse controls and the purpose of our paper.

2.1. **Pure continuous control problem.**

We consider a model of exploitation of natural resources in continuous time. In this kind of models the dynamics consists of an expression that corresponds to the natural evolution of the system to which is subtracted a rate of extraction. The natural evolution rate depends on the current state of the system. More precisely, the system evolves naturally according to $F(x)$ and there is the usual rate of continuous removal $\alpha(\cdot)$ which must be chosen from a suitable set $\{\alpha : \mathbb{R}^+ \to A \subset \mathbb{R}^+ \text{mesurable}\}$ with A compact. Suppose that these continuous controls provide an instantaneous profit $G(x(t))\alpha(t)$. The performance of these controls is measured by the reward over the entire time period, when the discount rate is $\rho > 0$. The associated continuous control problem is

$$\begin{cases} \dot{x}(t) = F\left(x(t)\right) - \alpha(t), \\ x(0) = x_0, \quad x(t) \geq 0, \end{cases} \tag{2.1}$$

$$\max_{\alpha \in \Lambda} J\left(x_0; \alpha\right) = \int_0^{+\infty} G\left(x(t)\right) \alpha(t) e^{-\rho t} dt. \tag{2.2}$$

The assumptions about the data (linearity of the dynamics and of the profit function with respect to the control) correspond to a singular problem. This means that the search of maximum leads to the Euler equation, which is not a differential one but an algebraic one in this situation. The solution of the algebraic equation could be interpreted as a state system level which is recommended to be achieved as quickly as possible and maintained over time.

Defining the function

$$C(x) = \left(GF\right)'(x) - \rho G(x), \tag{2.3}$$

the Euler equation associated to the problem is

$$C(x^*) = 0, \tag{2.4}$$

and it gives the steady state x^* of the problem.

If the steady state verifies the optimality condition:

$$(x^{*}-x)\,C(x) \geqq 0, \quad \forall x \geqq 0, \tag{2.5}$$

then, the optimal solution will be the most rapid approach path to this stationary state x^{*} provided that

$$0 \leq \ \alpha_{min} \leq \alpha(t) \leq \alpha_{max}, \quad \forall t. \tag{2.6}$$

This turnpike idea was first used by Samuelson [1949] who showed that an efficient expanding economy would be in the vicinity of a balanced equilibrium path for most of the time. Based on Miele [1962], who used the Green's theorem to solve lineal variational problems, Sethi [1977] has proved the turnpike property for a certain class of problems and Hartl and Feichtinger [1987], for example, have generalized the results to the non autonomous infinite horizon case. An application of this kind of model to natural resources extractions can also be seen in Clark [2005].

In this paper, unbounded controls are considered. In the following, we first present some results of pure impulse control. We then propose to widen the set of admissible controls to also allow for impulse controls in the mixed control model.

2.2. **Pure impulse control problem.**

Impulse controls consist in a sequence of moments, $0 \leqq \tau_1 < ... < \tau_i < \tau_{i+1} < \ldots$, and their corresponding amounts, ξ^{i}, $i = 1, 2, \ldots$. In these chosen time points part of the population is removed. The corresponding amount of the sequence produces abrupt changes in the evolution of the stock. The evolution of the system is

$$\begin{cases} \dot{x}(t) = F\left(x(t)\right), & \text{if} \quad t \neq \tau_i, \\ x\left(\tau_i^{+}\right) = x\left(\tau_i^{-}\right) - \xi^{i}, & \text{if} \quad t = \tau_i, \\ x(0) = x_0, & x(t) \geq 0. \end{cases} \tag{2.7}$$

Suppose that impulse controls provide an instantaneous profit $\bar{G}\left(x(\tau_i^-), \xi^i\right)$. The performance of these controls is measured, as before, by the reward over the entire time period, when the discount rate is $\rho > 0$, and the associated pure impulse control problem is:

$$\max_{\{\tau_i, \xi^i\}} J\left(x_0; \{\tau_i, \xi^i\}\right) = \sum_{i \in \mathbb{N}} e^{-\rho \tau_i} \, \bar{G}\left(x(\tau_i^-), \xi^i\right). \tag{2.8}$$

As shown in the literature, this problem has a unique cyclical solution if

$$\bar{G}(a, a-c) + \bar{G}(b, b-d) < \bar{G}(a, a-d) + \bar{G}(b, b-c), \quad d \leqq c \leqq b \leqq a. \tag{2.9}$$

See Erdlenbruch et al. [2013], for a detailed discussion of the types of solution to this problem.

As impulse controls are related with continuous "unbounded controls" we can ask for the limit of optimal profit function of the continuous control problem presented in section 2.1 when the upper bound of the control tends to infinite. This limit can be interpreted as the corresponding profit function for impulse controls. When the initial condition x_0 is greater than x^* MRAP optimal solution implies that we must harvest using $\alpha(t) = \alpha_{max}$. Increasing α_{max}, the approach path becomes steeper until describing a discrete harvesting process for an infinitely fast harvest capacity. The corresponding profit function is derived as follows:

$$\lim_{\alpha_{max} \to \infty} \int_0^{t_1} G(x(t)) \alpha_{max} dt, \quad \dot{x} = F(x) - \alpha_{max}, \quad x(0) = x_0, \; x(t_1) = x_1.$$

Let $x(t) = u$, $\dot{x} dt = du$, then

$$\lim_{\alpha_{max} \to \infty} \int_{x_0}^{x_1} G(x(t)) \frac{\alpha_{max}}{F(u) - \alpha_{max}} du = \int_{x_1}^{x_0} G(u) du. \tag{2.10}$$

As shown in Erdlenbruch et al. [2013], this is exactly the case, where condition 2.9 does not hold and the pure impulse control problem does not have an exact solution.

3. The model: the mixed control problem.

In this section, in order to verify Clark's conjecture we consider that the two types of harvest are allowed. According to these different types of behaviour, the evolution of the system is

$$\begin{cases} \dot{x}(t) = F\left(x(t)\right) - \alpha(t), & \text{if} \quad t \neq \tau_i, \\ x\left(\tau_i^{+}\right) = x\left(\tau_i^{-}\right) - \xi^i, & \text{if} \quad t = \tau_i, \\ x(0) = x_0, & x(t) \geq 0. \end{cases} \tag{3.1}$$

Let us call $\beta = \left(\alpha(\cdot), \left\{\tau_i, \xi^i\right\}_{i\in\mathbb{N}}\right)$ a mixed control which belongs to the admissible set of mixed controls $\mathcal{B}$. See appendix A for details. The performance of both kind of controls is measured by the total reward over the entire time period, when the discount rate is $\rho > 0$,

$$J\left(x_0;\beta\right) = \int_0^{+\infty} G(x(t))\alpha(t)e^{-\rho t}dt + \sum_{i\in\mathbb{N}} e^{-\rho\tau_i}\,\bar{G}\left(x\left(\tau_i^{-}\right), \xi^i\right). \tag{3.2}$$

The objective is to maximize profits and then the value function of the problem is

$$v(x_0) = \sup_{\beta\in\mathcal{B}} J\left(x_0;\beta\right). \tag{3.3}$$

3.1. **The Hamilton-Jacobi-Bellman equation for a mixed control problem.**

We present in this section the way to find an optimal solution of our mixed control problem. The value function (3.3) of our problem can be characterized by the Hamilton - Jacobi - Bellman equation, (HJB), associated to the problem. See Appendix A.

If the profit function of a mixed control is the unique solution of the associated (HJB) equation then this control is optimal.

The associated (HJB) equation is

$$\min\left\{\rho v(x) - H\left(x, Dv(x)\right); v(x) - \mathcal{M}v(x)\right\} = 0, \tag{3.4}$$

where

$$H\left(x, \lambda\right) = \sup_{a}\left\{G(x)a + \lambda\ \left[F(x) - a\right]\right\}, \tag{3.5}$$

and

$$\mathcal{M}(w)(x) = \sup_{\xi\in[0,x]}\left[w(x-\xi) + \bar{G}(x,\xi)\right], \tag{3.6}$$

for any real, bounded, uniformly continuous function w.

4. Specific functional forms of the dynamics and the profit functions.

To have the problem analytically tractable, in order to find an optimal solution or to prove when Clark's policy is not optimal, we are going to consider specific functional forms not only for the growth function of the natural resource but also for the two kinds of profit functions.

The proposed functional forms are common in the literature (see Erdlenbruch et al. [2013] for an overview). Instantaneous profits describe benefits from selling the harvest in a competitive market and costs of extraction. The natural growth function behaves like a logistic function, most often used in the literature, but has the following two advantages: first, it also allows for asymmetric growth patterns which can be observed in some natural resource stocks; second, it is computationally more tractable.

The following function F is considered:

$$F(x) = \begin{cases} rx, & \text{if} \quad x \leqq \dfrac{K}{1+r}, \\ K - x, & \text{if} \quad x > \dfrac{K}{1+r}, \end{cases} \tag{4.1}$$

where $r > 0$ represents a natural rate of the system and K, its carrying capacity. The growth of the resource is governed by a concave, piecewise linear, natural reproduction function, and restricted by a natural carrying capacity of the environment and then it can be considered $0 \leq x_0 \leq K$.

We consider the following instantaneous profit function

$$G(x(t))\,\alpha(t) = \left[p - \frac{c}{x(t)}\right]\alpha(t), \tag{4.2}$$

where the unit price p is a fixed constant and the cost of extraction per unit of time depends on $x(t)$ and takes the form $\frac{c}{x}$, with $c > 0$.

The total discounted profit of the continuous part of the control is therefore

$$J(x_0;\alpha) = \int_0^{+\infty} e^{-\rho t}\left[p - \frac{c}{x(t)}\right]\alpha(t)dt. \tag{4.3}$$

Inspired by (2.10), we consider the gain of the impulse extraction as

$$\begin{aligned} \bar{G}\left(x\left(\tau_i^-\right),\xi^i\right) &= \int_{x(\tau_i^-)-\xi^i}^{x(\tau_i^-)} G(\zeta)d\zeta = \int_{x(\tau_i^-)-\xi^i}^{x(\tau_i^-)} \left[p - \frac{\bar{c}}{\zeta}\right]d\zeta = \\ &= \left[p\xi^i - \bar{c}\ln\left(\frac{x\left(\tau_i^-\right)}{x\left(\tau_i^-\right)-\xi^i}\right)\right], \quad 0 \leqq \xi \leqq x(\tau_i). \end{aligned} \tag{4.4}$$

where $\bar{c} > 0$ represents the cost of instantaneous extraction, which is also considered constant. This function is homogeneous, i.e., the jump times do not appear explicitly. Note that when $\bar{c} = c$, we can say that the profit functions are equivalent in the sens that they produce the same gain.

Therefore, from (4.3) and (4.4), our mixed control problem is:

$$\begin{aligned} J(x_0;\beta) = &\int_0^{+\infty} e^{-\rho t}\left(p - \frac{c}{x(t)}\right)\alpha(t)dt + \\ &+ \sum_{i\in\mathbb{N}} e^{-\rho\tau_i}\left[p\xi^i - \bar{c}\ln\left(\frac{x\left(\tau_i^-\right)}{x\left(\tau_i^-\right)-\xi^i}\right)\right]. \end{aligned} \tag{4.5}$$

Remark 1. *We suppose that*

$$K > \frac{c}{p}. \tag{4.6}$$

As we are going to see later if this assumption is not verified, there is no positive steady state solution.

4.1. **The steady state for the pure continuous control problem.**

Our idea is to know when the optimal solution of our mixed control problem is to jump at time $t = 0$ to the steady state of the continuous control problem and stay in it forever. We are hence going to identify this steady state.

Depending on the relationship between the parameters involved in the model, the Euler equation leads to different cases but all of them with a unique stable steady state. This uniqueness avoids the need to choose which is the optimal steady state.

Notation 1. *These following values will be used in the analysis:*

$$\begin{aligned} \nu_1 &= \lim_{x \to \frac{K}{1+r}^-} (FG)'(x) = rp, \\ \nu_2 &= \lim_{x \to \frac{K}{1+r}^+} (FG)'(x) = \frac{c(1+r)^2}{K} - p, \\ \nu_3 &= \rho G\left(\frac{K}{r+1}\right) = \rho\left(p - \frac{c(1+r)}{K}\right), \\ \rho_1 &= \frac{c(1+r)^2 - pK}{pK - c(1+r)}, \quad \rho_2 = \frac{rpK}{pK - c(1+r)}. \end{aligned} \tag{4.7}$$

Proposition 1. *Existence and uniqueness of steady state.*

<*i*> *If*

$$\frac{\nu_2}{p} < r \quad \wedge \quad \rho > \rho_2, \tag{4.8}$$

then there exists a unique positive steady state

$$x^* = \frac{c}{p}\frac{\rho}{\rho - r}, \tag{4.9}$$

which verifies

$$\frac{c}{p} < x^* < \frac{K}{r+1}. \tag{4.10}$$

<*ii*> *If*

(a)-

$$\frac{\nu_2}{p} < r \quad \wedge \quad \rho < \rho_1, \tag{4.11}$$

or if

(b)-

$$r \leqq \frac{\nu_2}{p}, \quad \forall \rho > 0, \tag{4.12}$$

then there exists a unique positive steady state

$$x^* = \frac{\rho c + \sqrt{\rho^2 c^2 + 4pc\,(\rho+1)\,K}}{2p\,(\rho+1)}, \tag{4.13}$$

which verifies

$$\frac{K}{r+1} < x^* < \sqrt{\frac{Kc}{p}}. \tag{4.14}$$

See the proof of this proposition in Appendix B.

Remark 2. *Thanks to the working assumption* (4.6), $\rho_1 \leqq \rho_2$ *and then the* $< i >$*-case and the* $< ii >$ (a) *one are not empty.*

Remark 3. *It is worth mentioning that under relations among parameters not considered in the previous proposition, i.e., when*

$$\nu_2 < \nu_1 \wedge \rho_1 \leqq \rho \leqq \rho_2, \tag{4.15}$$

the Euler equation does not have a solution. From their definitions,

$$\nu_2 < \nu_3 < \nu_1. \tag{4.16}$$

Since $\nu_2 < \nu_1$*, it results that* $\rho_1 \neq \rho_2$ *and* ρ *could be between them. In this case,* $\dfrac{K}{1+r}$ *is proposed as a possible "steady state".*

5. Resolution of the mixed control problem.

5.1. **When an instantaneous jump to the steady state is the optimal solution.**

This section will analyse the case where the profit functions are related in a particular manner. In our model, this means that $c = \bar{c}$. This will be a sufficient condition for the following mixed control to be optimal.

$$(5.1)\quad \beta^* = \begin{cases} \left(\alpha(t) = F\left(x^*\right), t > 0; \left\{\tau_1 = 0, \xi^1 = x_0 - x^*\right\}\right), & x_0 > x^*, \\ \left(\alpha(t) = F\left(x^*\right), t \geqq 0\right), & x_0 = x^*, \\ \left(\alpha(t) = F\left(x^*\right), t > \tau(x_0)\right), & x_0 < x^*, \end{cases}$$

where $\tau(x_0)$ is the time that the dynamics needs to reach the x^* level from the initial level x_0, when $x_0 < x^*$.

Again, the main idea is to achieve the desired level x^* as quickly as possible (turnpike property). This mixed control jumps to that level at the initial instant $t = 0$, when $x_0 > x^*$. If $x_0 < x^*$, it is necessary to wait for the system to evolve naturally and to reach the steady state.

Proposition 2. *For the optimality problem of maximizing* (4.5)*, following the dynamics* (3.1)*, with evolution function* (4.1)*, and value function* (3.3)*, with* $c = \bar{c}$*, the mixed control* β^* *defined in* (5.1) *is an optimal one.*

See the proof of this proposition in Appendix C.

5.2. **When an instantaneous jump to the steady state is not the optimal solution.**

5.2.1. *Case:* $\bar{c} > c$ *.*

When the continuous extraction cost is lower than the instantaneous one, the proposed control is not optimal.

The idea is to compare it with another which uses a constant continuous control to reach the steady state.

Remember that the former control with our functional forms is

$$\beta^* = \begin{cases} \left(\alpha(t) = rx^*, t > 0; \left\{t_1 = 0, \xi^1 = x_0 - x^*\right\}\right), & x_0 > x^*, \\ \left(\alpha(t) = rx^*, t \geqq 0\right), & x_0 = x^*, \\ \left(\alpha(t) = \begin{cases} 0, & 0 \leqq t < \tau(x_0) \\ rx^*, & t \geqq \tau(x_0) \end{cases}\right), & x_0 < x^*, \end{cases} \tag{5.2}$$

where $\tau(x_0)$ is the time that the dynamics needs to reach the x^* level from the initial level x_0, when $x_0 < x^*$.

A new continuous control which depends on a constant value $a > 0$ can be considered:

$$\hat{\beta}_a = \begin{cases} \left(\alpha(t) = \begin{cases} a, & 0 \leqq t < \hat{\tau}(a) \\ rx^*, & t \geqq \hat{\tau}(a). \end{cases}\right), & x_0 > x^*, \\ \left(\alpha(t) = rx^*, t \geqq 0\right), & x_0 = x^*, \\ \left(\begin{cases} 0, & 0 \leqq t < \tau(x_0) \\ rx^*, & t \geqq \tau(x_0) \end{cases}\right), & x_0 < x^*, \end{cases} \tag{5.3}$$

where

$$\hat{\tau}(a) = \frac{1}{r} \ln \left(\frac{a - rx^*}{a - rx_0} \right),$$

the time required by the system to descend to level x^*.

Now let us focus on the case where $x_0 > x^*$. With this control (5.3), if $x_0 > x^*$, the resource does not jump to the steady state. Instead, the stock descends to x^* thanks to the constant continuous control equal to a.

The resulting trajectory is

$$x(t) = \frac{a}{r} + e^{rt}\left(x_0 - \frac{a}{r}\right).$$

The benefits obtained with the two previous controls are respectively

$$J(x_0;\beta^*) = \int_{x^*}^{x_0}\left(p - \frac{\bar{c}}{\nu}\right)d\nu + \int_0^{\hat{\tau}(a)}\left(p - \frac{c}{x^*}\right)rx^*e^{-\rho t}dt + \int_{\hat{\tau}(a)}^{\infty}\left(p - \frac{c}{x^*}\right)rx^*e^{-\rho t}dt, \tag{5.4}$$

$$J\left(x_0;\hat{\beta}\right) = \int_0^{\hat{\tau}(a)}\left(p - \frac{c}{x(t)}\right)ae^{-\rho t}dt + \int_{\hat{\tau}(a)}^{\infty}\left(p - \frac{c}{x^*}\right)rx^*e^{-\rho t}dt. \tag{5.5}$$

Analysing these benefits as functions of the constant a, it can be seen that there is a level $\bar{a}$ that produces the same gain as jumping when starting. Furthermore, if a higher constant control is used, the gain increases.

Therefore,

$$\exists \bar{a} = \bar{a}(x_0)/ \quad J(x_0,\beta^*) = J\left(x_0,\hat{\beta}_{\bar{a}}\right) \quad \wedge$$
$$J\left(x_0,\hat{\beta}_a\right) > J(x_0,\beta^*), \quad \forall a > \bar{a}.$$

Below numerical examples are displayed, where $h = \dfrac{\rho}{r}$ indicates the relationship between parameters r, the natural growth rate of the resource, and ρ, the discount rate of the model.

Parameter	value
K	3200
c	70
p	3/5
ρ	1/2
r	1/4
h	2
x^*	$233,\hat{3}$

Using these values, Figure 1 on page 17 shows always a positive difference between the profits as a function of the initial value, with different instantaneous cost constants. A positive difference indicates that it is better to use the constant control a to reach the x^* value than to jump at $t = 0$ to this level.

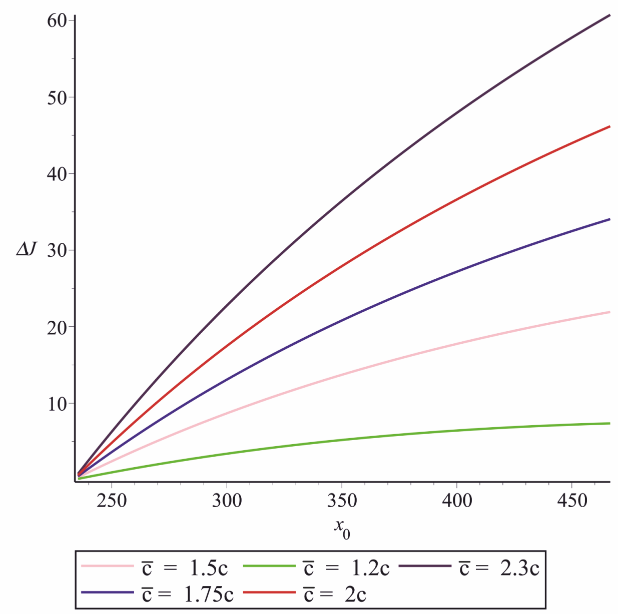

FIGURE 1. Difference between the profits, function of the initial value.

Figure 2 on page 17 shows those positive differences as functions of the constant continuous control $\alpha(t) \equiv a$, again with different instantaneous cost constants. The improvement achieved using a continuous control a grows with a.

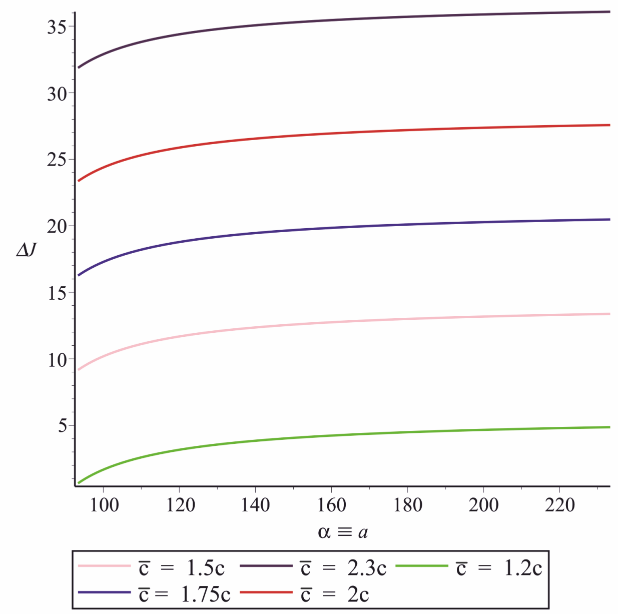

FIGURE 2. Difference between the profits, function of a.

We can then state the following result

Proposition 3. *When $x_0 \geqq x^*$ and $\bar{c} > c$, descending to the level x^* using a continuous constant control greater than a^* is better than using the proposed control.*

5.2.2. *Case: $\bar{c} < c$.*

Let us focus first on the case $x(0) = x^*$ and compare the gain of an initial extraction with the gain of using β^*, (5.2).

If $x(0) = x^*$, following the proposed control β^*, the system will remain at that level, which is obtained by using the continuous constant control equal to rx^*. But if a jump of size ξ is produced at $t = 0$, the system needs $\tau(x^* - \xi)$ time to reach the level x^* again.

In order to contrast both situations, we will compare the profits up to $\tau(x^* - \xi)$, from where both controls produce the same situation, by using the following functions.

To jump ξ at the beginning and then wait until $\tau(x^* - \xi)$ produces this partial benefit:

$$J_1(\xi) = p\xi - \bar{c}\ln\left(\frac{x^*}{x^* - \xi}\right), \tag{5.6}$$

meanwhile, maintaining $x(t) \equiv x^*$ until $\tau(x^* - \xi)$ produces this partial benefit:

$$J_2(\xi) = \left(p - \frac{c}{\xi}\right)\frac{x^*}{h}\left(1 - \left(\frac{x^* - \xi}{x^*}\right)^h\right). \tag{5.7}$$

The steady state level corresponding to a continuous benefit with a cost $\bar{c}$ would be

$$x^*_{\bar{c}} = \frac{\bar{c}h}{p(h-1)}. \tag{5.8}$$

Since $0 < \bar{c} < c$, it can be considered this relation:

$$\bar{c} = sc, \quad 0 < s < 1. \tag{5.9}$$

The benefit of jumping directly to the x^* level can be compared with the benefit of jumping to this new $x^*_{\bar{c}}$ level and then wait until the systems evolves to x^*, since $x^*_{\bar{c}} < x^*$.

It is only necessary to analyse the case where $x_0 = x^*$, for which is the case of our interest.

A jump to the level $x^*_{\bar{c}}$ and then wait produces

$$\begin{aligned} J_1\left(x^* - x^*_{\bar{c}}\right) &= p\left(x^* - x^*_{\bar{c}}\right) - \bar{c}\ln\left(\frac{x^*}{x^*_{\bar{c}}}\right) = \\ &= \left(c - \bar{c}\right)\frac{h}{h-1} - \bar{c}\ln\left(\frac{c}{\bar{c}}\right) = \\ &= \left(1 - s\right)c\frac{h}{h-1} + cs\ln(s). \end{aligned} \tag{5.10}$$

To maintain x^* level until the time $\tau\left(x^* - \xi\right)$ produces

$$\begin{aligned} J_2\left(x^* - x^*_{\bar{c}}\right) &= \left(p - \frac{c}{x^*}\right)\frac{x^*}{h}\left(1 - \left(\frac{x^*_{\bar{c}}}{x^*}\right)^h\right) = \\ &= \left(p - \frac{c}{x^*}\right)\frac{x^*}{h}\left(1 - \left(\frac{\bar{c}}{c}\right)^h\right) = \\ &= \frac{c}{(h-1)\,h}\left(1 - s^h\right). \end{aligned} \tag{5.11}$$

Since[2]

$$\begin{aligned} &\frac{d}{ds}\left[h^2\left(1-s\right) + s + h\left(h-1\right)\ln(s) - 1 + s^h\right] = \\ &= h\left[(h-1)\ln(s) + s^{h-1} - 1\right] < 0, \forall 0 < s < 1, \end{aligned} \tag{5.12}$$

and

$$\lim_{s\to 1}\left[h^2\left(1-s\right) + s + h\left(h-1\right)\ln(s) - 1 + s^h\right] = 0, \tag{5.13}$$

from (5.11) and (5.10), $\forall 0 < s < 1$, the resulting difference between the two partial benefits is

$$\begin{gathered} J_1\left(x^* - x^*_{\bar{c}}\right) - J_2\left(x^* - x^*_{\bar{c}}\right) = \\ = \frac{c}{h\left(h-1\right)}\left[h^2\left(1-s\right) + s + h\left(h-1\right)\ln(s) - 1 + s^h\right] > 0. \end{gathered} \tag{5.14}$$

[2] $1 - x > \ln x$ when $0 < x < 1$.

We can then state that

Proposition 4. *When* $x_0 \geqq x^*$ *and* $\bar{c} < c$*, jumping at the beginning to the level* $x^*_{\bar{c}} \leq x^*$ *is better than using the proposed control.*

Remark 4. *A case to be considered is when* $\bar{c} = 0$ *since* $x^*_{\bar{c}} = 0$*. In this case, the recommended instantaneous extraction* $\xi = x_0 - x^*_{\bar{c}}$ *at* $t = 0$ *leads to extinction but it is better than the mixed option* β^* *due to the zero instantaneous cost.*

6. Conclusion

In this paper, we study a mixed optimal control problem in which both continuous controls and impulse controls are admissible. The optimal solution of this problem can be characterized via the Hamilton-Jacobi-Bellman equation. However, the resolution of this equation can be cumbersome and in most cases only numerical solutions are possible. This is why we consider an example with particular functional forms.

Building on the solutions of the pure continuous control and the pure impulse control problem, we propose a candidate for the optimal solution of the mixed control problem. We prove that this candidate verifies the Hamilton-Jacobi-Bellman equation. As conjectured by Clark, jumping to the steady state and then staying there is one possible optimal solution, but we show that the profit functions of the continuous control and impulse control sub-models need to be related in a particular manner to reach such solution. Although the above results were obtained with particular functional forms, we think that it will be possible to prove the optimality of Clark's policy for general functional forms of growth and profit functions under the condition that the latter are linked in the way we indicate in this paper. This is our "conjecture" and it is work in progress.

In a wider sense, our work stresses the importance of the link between the structure of the model, the model assumptions and the functional forms of the model. When changing the assumptions on the harvest variable, we also need to change assumptions on the form of associated benefit and cost functions. This procedure is not always followed in the literature on resource economics. Moreover, our analysis also reminds that switching from models with bounded controls to models with unbounded controls is not trivial.

Appendix A. General model

The model considered can be posed in the following general form. A system governed by a differential equation is considered:

$$\text{(A.1)} \qquad \begin{cases} \dfrac{dx}{ds}(s) = f\left(x(s), \alpha(s)\right) - \sum\limits_{i \in \mathbb{N}} \delta_{(s-\tau_i)} \xi^i, \\ x(0) = x_0, \end{cases}$$

where $\beta := \left(\alpha, \left\{\tau_i, \xi^i\right\}_{i \in \mathbb{N}}\right)$ can be chosen from a suitable set $\mathcal{B}$ and $\delta_{(s-\tau_i)}$ is the Dirac function. This control is assessed by the criterion

$$\text{(A.2)} \qquad J\left(x_0; \beta\right) = \int_0^{+\infty} l\left(x(s), \alpha(s)\right) e^{-\rho s} ds + \sum_{i \in \mathbb{N}} e^{-\rho \tau_i} \bar{l}\left(x\left(\tau_i^-\right), \xi^i\right).$$

The function $x(s)$ represents the evolution of the system starting at x_0, when it is subjected to continuous control α, and to instantaneous control of size ξ^i at times τ_i. Both controls produce benefits, respectively represented by l and $\bar{l}$.

A particular case of what it is presented in (A.1)-(A.2) is considered in Bardi and Capuzzo-Dolcetta [1997]. This problem generalizes the one presented in that book, since here the impulse controls and also the impulse profit function both depend on the state of the system.

As usual, the problem is to find (if any) a control that maximizes the criterion.

The value function of the problem is

$$\text{(A.3)} \qquad v(x) = \sup_{\beta \in \mathcal{B}} J\left(x_0; \beta\right).$$

The following hypotheses are assumed:

<S1> $x(s), x_0 \in \mathbb{R}^n_+, \ \forall s \geqq 0.$

<S2> Let $\beta = \left(\alpha(s), s \geqq 0, \left\{\tau_i, \xi^i\right\}_{i \in \mathbb{N}}\right) \in \mathcal{B}$ a mixed control.

<S3> $\xi^i \in K\left(x(\tau_i)\right) \subset \mathbb{R}^n_+$, $K : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m$ is a lower semicontinuous compact multivalued map.

<S4> $\tau_1 \geqq 0$ and $\exists\ t^* > 0$ such that $\tau_{i+1} \geqq \tau_i + t^*,\ \forall i \in \mathbb{N}$.

<S5> $\alpha : \mathbb{R}^+ \to A \subset \mathbb{R}^m$ is measurable and A compact.

<S6> $\exists\ N > 0$ such that $|f(x,a)| \leqq N$, $\forall x \in \mathbb{R}^n$, $\forall a \in A$, f is continuous and $f(\cdot, a)$ is one-sided Lipschitz.

<S7> $\exists\ M > 0$ such that $|l(x,a)| \leqq M$, $\forall x \in \mathbb{R}^n$, $\forall a \in A$ and $l(\cdot,a)$ admits a global modulus of continuity, ω_l, i.e.,

$$|l(x,a) - l(y,a)| \leqq \omega_l(|x-y|), \qquad \forall a \in A.$$

<S8> $\exists\ C > 0$ such that $\sup_{\xi \in K(x)} \left|\bar{l}(x,\xi)\right| \leqq C$, $\forall x \in \mathbb{R}^n_+$ and $\bar{l}$ admits a modulus of continuity ω_c, i.e.,

$$\left|\bar{l}(x,\xi) - \bar{l}(y,\nu)\right| \leqq \omega_c(\|(x,\xi) - (y,\nu)\|), \qquad \xi \in K(x),\ \nu \in K(y).$$

Following the guidelines used in the books [Bardi and Capuzzo-Dolcetta, 1997, Barles, 1994] the value function of this problem can be represented as the solution of a Bellman type equation (cf. Alvarez et al. [preprint]). More precisely,

Theorem 1.

v defined in (A.3) *is the unique viscosity solution of*

$$\min\{\rho\, v(x) - H(x, Dv(x)); v(x) - \mathcal{M}v(x)\} = 0, \tag{A.4}$$

where

$$H(x,\lambda) = \sup_{a \in A}\{l(x,a) + \lambda\, f(x,a)\}, \tag{A.5}$$

and

$$\mathcal{M}(w)(x) = \sup_{\xi \in K(x)} \left[w(x-\xi) + \bar{l}(x,\xi)\right], \quad w \in \mathcal{BUC}(\mathbb{R}^n). \tag{A.6}$$

Appendix B. Steady state. Proof of Proposition 1

Proof. (i)- Working in $0 < x < \dfrac{K}{1+r}$ and rewriting the Euler equation (2.4),

$$\rho\left(p - \frac{c}{x}\right) = \left(p - \frac{c}{x}\right) r + \frac{c}{x^2} r x.$$

When $\rho \neq r$,

$$x^* = \frac{c\rho}{p(\rho - r)}.$$

Then x^* will be the unique positive solution of the Euler equation in this case iff $\rho > r$. From (4.7), it results

$$\nu_2 < \nu_1 < \nu_3. \tag{B.1}$$

Since $\rho > \rho_2$,

$$\rho > \frac{r\dfrac{K}{1+r}}{\dfrac{K}{1+r} - \dfrac{c}{p}}. \tag{B.2}$$

Besides, since $\nu_2 < \nu_1$,

$$\frac{K}{1+r} - \frac{c}{p} > 0. \tag{B.3}$$

From (B.2) and (B.3), $\rho > r$. Therefore

$$x^* > 0. \tag{B.4}$$

Moreover,

$$D\left[\rho G\right](x) = \frac{\rho c}{x^2} > 0,\ \forall x \Rightarrow \rho G \text{ is increasing.} \tag{B.5}$$

From (B.5), it is verified

$$\frac{c}{p} < x^* < \frac{K}{r+1}. \tag{B.6}$$

Verifying the turnpike optimality:

$$C(x) = p(r-\rho) + \frac{\rho c}{x},$$

$$(x_1^* - x)\left[p(r-\rho) + \frac{\rho c}{x}\right] \geqq 0, \quad \forall x > 0.$$

Then, this x^* steady state verifies the condition (2.5).

(ii)- Working in $\frac{K}{1+r} < x < K$, rewriting (2.4),

$$\rho\left(p - \frac{c}{x}\right) = \left(p - \frac{c}{x}\right)(-1) + \frac{c}{x^2}(K-x),$$

$$p(\rho+1)x^2 - \rho c x - Kc = 0.$$

Then,

(B.7)
$$x^* = \frac{\rho c + \sqrt{\rho^2 c^2 + 4pc(\rho+1)K}}{2p(\rho+1)}.$$

This x^* will be the unique positive solution of the Euler equation in this case. When the parameters verify the conditions (4.11), from (4.7), it results

(B.8)
$$\nu_3 < \nu_2 < \nu_1.$$

Since $\nu_3 < \nu_2$, from (B.8),

(B.9)
$$x^* > \frac{K}{r+1}.$$

When the parameters verify the conditions (4.12), from (4.7), with $\rho > 0$, it results

(B.10)
$$\nu_3 \leqq 0.$$

Since $\nu_3 \leqq 0$, from (B.10),

(B.11)
$$x^* > \frac{K}{r+1}.$$

Moreover, working with $x > \dfrac{K}{r+1}$,

$$D\left[FG\right](x) = \frac{Kc}{x^2} - p = 0 \Leftrightarrow x = \sqrt{\frac{Kc}{p}}. \tag{B.12}$$

Since $D\left[FG\right]$ is decreasing, from (B.12), a positive steady state x^* will verified

$$x^* < \sqrt{\frac{Kc}{p}}. \tag{B.13}$$

Verifying the turnpike optimality:

$$\begin{aligned} C(x) &= \left(\frac{c}{x} - p\right)(\rho + 1) + c\frac{K - x}{x^2} = \\ &= \frac{\rho c}{x} + \frac{cK}{x^2} - p\left(\rho + 1\right). \end{aligned}$$

$$C'(x) = -c\frac{\rho x + 2K}{x^3} < 0.$$

Therefore,

$$\left(x^* - x\right) C(x) \geqq 0, \forall x \geqq 0.$$

When the parameters of the model verify item (a) or item (b), the steady state solution verifies the condition (2.5).

□

Appendix C. Mixed optimal control. Proof of Proposition 2

Proof. Different cases will be considered, depending on the relationship between parameters.

(A) - First, consider the case where $\nu_2 < \nu_1$ and $\rho > \rho_2$, as in (4.8). Then, it results

$$x^* = x_1^* = \frac{c}{p}\frac{\rho}{\rho - r}.$$

Rewriting (5.1),

$$\text{(C.1)} \qquad \beta^* = \begin{cases} \left(\alpha(t) = rx^*, t > 0; \left\{\tau_1 = 0, \xi^1 = x_0 - x^*\right\}\right), & x_0 > x^*, \\ \left(\alpha(t) = rx^*, t > \tau(x_0)\right), & x_0 \leqq x^*, \end{cases}$$

The profit function corresponding to the proposed control (C.1) is

(C.2)

$$J(x_0; \beta^*) = \begin{cases} \int\limits_{x_1^*}^{x_0} \left(p - \frac{\bar{c}}{\nu}\right) d\nu) + \int\limits_{0}^{+\infty} \left(p - \frac{c}{x_1^*}\right) r x_1^* e^{-\rho t} dt, & x_0 > x_1^*, \\ \int\limits_{\tau(x_0)}^{+\infty} \left(p - \frac{c}{x_1^*}\right) r x_1^* e^{-\rho t} dt, & x_0 \leqq x_1^*. \end{cases}$$

In this case, the time $\tau(x_0)$ is

$$\text{(C.3)} \qquad \tau(x_0) = \frac{1}{r} \ln\left(\frac{x_1^*}{x_0}\right), \quad x_0 \leqq x_1^*.$$

Then, rewriting with $h = \dfrac{\rho}{r}$,

$$\text{(C.4)} \quad J(x_0; \beta^*) = \begin{cases} p\,(x_0 - x_1^*) - \bar{c} \ln\left(\frac{x_0}{x_1^*}\right) + \left(p - \frac{c}{x_1^*}\right) \frac{x_1^*}{h}, & x_0 > x_1^* \\ \left(p - \frac{c}{x_1^*}\right) \frac{x_0^h}{h} (x_1^*)^{1-h}, & x_0 \leqq x_1^*. \end{cases}$$

The idea is to verify that the benefit function (C.2) related to the proposed control (5.1) satisfies the Bellman equation (3.4). The Hamiltonian is

$$\text{(C.5)} \qquad \begin{aligned} H(x, \lambda) &= \sup_a \{l(x,a) + \lambda f(x,a)\} = \\ &= \sup_{a \geqq 0} \left\{\left[p - \frac{c}{x}\right] a + \lambda\,(rx - a)\right\} = \\ &= \sup_{a \geqq 0} \left\{\lambda r x + \left[p - \frac{c}{x} - \lambda\right] a\right\} = \\ &= \begin{cases} \lambda r x, & p - \frac{c}{x} - \lambda \leqq 0, \\ +\infty, & p - \frac{c}{x} - \lambda > 0. \end{cases} \end{aligned}$$

Analysing the conditions in (C.5),

$$\text{(C.6)}\quad p-\frac{c}{x_0}-DJ\left(x_0;\beta^*\right)=\begin{cases} p-\dfrac{c}{x_0}-p+\dfrac{\bar{c}}{x_0}, & x_0>x_1^*,\\ p-\dfrac{c}{x_0}-\left[p-\dfrac{c}{x_1^*}\right]\left(\dfrac{x_0}{x_1^*}\right)^{h-1}, & x_0<x_1^*.\end{cases}$$

Rewriting (C.6),

$$\text{(C.7)}\quad p-\frac{c}{x_0}\leqq\left[p-\frac{c}{x_1^*}\right]\left(\frac{x_0}{x_1^*}\right)^{h-1}\Leftrightarrow\left(p-\frac{c}{x_0}\right)x_0^{1-h}\leqq\left[p-\frac{c}{x_1^*}\right]\left(x_1^*\right)^{1-h}.$$

Since $\left(p-\dfrac{c}{x}\right)x^{1-h}$ is increasing in x, the previous inequality is verified.

Therefore,

$$\text{(C.8)}\qquad H\left(x_0,DJ\left(x_0;\beta^*\right)\right)=DJ\left(x_0;\beta^*\right)\ r\ x_0,\quad x_0\geqq 0,$$

taking into account the assumption that $\bar{c}=c$.

Remembering the definition, from (3.6),

$$\begin{aligned}\mathcal{M}u(x) &= \sup_{\xi\in K(x)}\left[u\left(x-\xi\right)+\bar{l}\left(x,\xi\right)\right]=\\ &= \sup_{0\leqq\xi\leqq x}\left[u\left(x-\xi\right)+p\xi-\bar{c}\ln\left(\frac{x}{x-\xi}\right)\right].\end{aligned}$$

The functional $\mathcal{M}$ evaluated at the benefit function of using the β^* control is

$$\mathcal{M}J\left(x_0;\beta^*\right)=$$

$$=\sup_{0\leqq\xi\leqq x_0}\begin{cases} p\left(x_0-x_1^*\right)+\left[p-\dfrac{c}{x_1^*}\right]\dfrac{x_1^*}{h}-\bar{c}\ln\left(\dfrac{x_0}{x_1^*}\right), & x_0-x_1^*\geqq\xi,\\ \left[p-\dfrac{c}{x_1^*}\right]\dfrac{\left(x_1^*\right)^{1-h}}{h}\left(x_0-\xi\right)^h+p\xi+\bar{c}\ln\left(\dfrac{x_0-\xi}{x_0}\right), & x_0-x_1^*<\xi.\end{cases}$$

When $x_0<x_1^*$, $x_0-x_1^*<0$ and it is only considered $0\leqq\xi\leqq x_0$.

$$\begin{aligned}&\sup_{0\leqq\xi\leqq x_0}\left[p-\frac{c}{x_1^*}\right]\frac{\left(x_1^*\right)^{1-h}}{h}\left(x_0-\xi\right)^h+p\xi+\bar{c}\ln\left(\frac{x_0-\xi}{x_0}\right)=\\ &\qquad=\left[p-\frac{c}{x_1^*}\right]\frac{x_1^*}{h}\left(\frac{x_0}{x_1^*}\right)^h.\quad(\xi=0)\end{aligned}$$

Since $c = \bar{c}$,

$$\frac{d}{d\xi}(\cdot) = p - \left[p - \frac{c}{x_1^*}\right]\left(\frac{x_0 - \xi}{x_1^*}\right)^{h-1} - \frac{\bar{c}}{x_0 - \xi} < 0.$$

When $x_0 > x_1^*$, $x_0 - x_1^* > 0$ so $0 \leqq \xi \leqq x_0 - x_1^*$ and also $x_0 - x_1^* \leqq \xi \leqq x_0$.

Then,

(C.9)

$$\mathcal{M}J\left(x_0;\beta^*\right) = \begin{cases} \left[p - \dfrac{c}{x_1^*}\right]\dfrac{x_1^*}{h}\left(\dfrac{x_0}{x_1^*}\right)^h, & x_0 < x_1^*, \\ p\left(x_0 - x_1^*\right) + \left[p - \dfrac{c}{x_1^*}\right]\dfrac{x_1^*}{h} - \bar{c}\ln\left(\dfrac{x_0}{x_1^*}\right), & x_0 > x_1^*. \end{cases}$$

Rewriting,

$$\text{(C.10)} \quad \mathcal{M}J\left(x_0;\beta^*\right) = \begin{cases} \dfrac{c}{h(h-1)}\left(\dfrac{x_0}{x_1^*}\right)^h, & x_0 < x_1^*, \\ p\left(x_0 - x_1^*\right) + \dfrac{c}{h(h-1)} - \bar{c}\ln\left(\dfrac{x_0}{x_1^*}\right), & x_0 > x_1^*. \end{cases}$$

Remembering the Bellman equation, (3.4), from (C.8) and (C.4), the resulting left part is

$$\begin{aligned} &\rho J\left(x_0;\beta^*\right) - DJ\left(x_0;\beta^*\right)\ r\ x_0 = \\ &= r\left[hJ_{\beta^*}(x_0) - x_0 DJ_{\beta^*}(x_0)\right], \end{aligned} \qquad \forall x_0 \geqq 0.$$

Now,

$$hJ\left(x_0;\beta^*\right) - x_0 DJ\left(x_0;\beta^*\right) =$$

$$= \begin{cases} hp\left(x_0 - x_1^*\right) - h\bar{c}\ln\left(\dfrac{x_0}{x_1^*}\right) + \left[p - \dfrac{c}{x_1^*}\right]x_1^* - px_0 + \bar{c}, & x_0 > x_1^*, \\ \left[p - \dfrac{c}{x_1^*}\right]x_1^*\left(\dfrac{x_0}{x_1^*}\right)^h - \left[p - \dfrac{c}{x_1^*}\right]\left(x_1^*\right)^{1-h}x_0^h, & x_0 < x_1^*. \end{cases}$$

$$= \begin{cases} p\left(h-1\right)\left(x_0 - x_1^*\right) + \bar{c} - h\bar{c}\ln\left(\dfrac{x_0}{x_1^*}\right) - c, & x_0 > x_1^*, \\ 0, & x_0 < x_1^*. \end{cases}$$

It results

$$p\,(h-1)\,(x_0 - x_1^*) - hc\ln\left(\frac{x_0}{x_1^*}\right) \geqq 0 \Leftrightarrow x_0 - x_1^* \geqq x_1^* ln\left(\frac{x_0}{x_1^*}\right).$$

But

$$\frac{x_0}{x_1^*} \geqq \left[\ln\left(\frac{x_0}{x_1^*}\right) + 1\right], \quad \forall \frac{x_0}{x_1^*} \geqq 1.$$

Therefore,

(C.11) $$\rho\ J\,(x_0;\beta^*) - DJ\,(x_0;\beta^*)\ r\ x_0 \geqq 0.$$

The right part of (3.4), from (C.10) and (C.4), is

(C.12) $$J\,(x_0;\beta^*) - \mathcal{M}J\,(x_0;\beta^*) = 0, \quad \forall x_0 \geqq 0.$$

Therefore, from (C.11) and (C.12), $J_{\beta^*}(x_0)$ verifies the Bellman equation (3.4) corresponding to this problem.

The proposed control is an optimal one since its profit function (C.2) is the solution of (3.4) working within the constraints of this case.

(B) - When the parameters verify the relations of (4.15), $\nu_2 < \nu_1 \wedge \rho_1 \leqq \rho \leqq \rho_2$, then

(C.13) $$x^* = x_2^* = \frac{K}{1+r}.$$

As in the previous case, the mixed control (5.1) is the one proposed.

Following the ideas exposed in the previous case, it can be seen that this control results optimal for the problem.

(C) - Working with parameters that verify the conditions of (4.11), the steady state is

(C.14) $$x^* = x_3^* = \frac{\rho c + \sqrt{\rho^2 c^2 + 4pc\,(\rho+1)\,K}}{2p\,(\rho+1)}.$$

If the initial level is below x_3^*, the system needs a time to reach that level by itself. Let τ_3 be that time:

$$\tau_3(x_0) = \begin{cases} \ln\left(\dfrac{K - x_0}{K - x_3^*}\right), & \dfrac{K}{r+1} \leqq x_0 \leqq x_3^*, \\ \ln\left(\dfrac{K\frac{r}{r+1}}{K - x_3^*}\right) + \dfrac{1}{r}\ln\left(\dfrac{K}{x_0\,(1+r)}\right), & 0 < x_0 < \dfrac{K}{1+r}. \end{cases} \tag{C.15}$$

For this case, rewriting (5.1), the proposed control is β_3:

$$\beta_3 = \begin{cases} \alpha(t) = K - x_3^*, t > 0;\ \left\{\tau_1 = 0, \xi^1 = x_0 - x_3^*\right\}, & x_0 > x_3^* \\ \alpha(t) = K - x_3^*, t \geqq 0, & x_0 = x_3^*, \\ \alpha(t) = \begin{cases} 0, & 0 \leqq t < \tau_3\,(x_0)\,, \\ K - x_3^*, & t \geqq \tau_3\,(x_0)\,, \end{cases} & 0 < x_0 < x_3^*. \end{cases} \tag{C.16}$$

(D) - Working with parameters that verify the conditions of (4.12), the steady state x_4^* is the same as the previous one which was calculated on (C.14).

If the initial level is below x_4^*, the system needs τ_3 to reach that level by itself as was shown in (C.15).

For this case, the proposed control β^* takes the form of β_4:

$$\beta_4 = \begin{cases} \alpha(t) = K - x_4^*, t > 0;\ \left\{\tau_1 = 0, \xi^1 = x_0 - x_4^*\right\}, & x_0 > x_4^* \\ \alpha(t) = K - x_4^*, t \geqq 0, & x_0 = x_4^*, \\ \alpha(t) = \begin{cases} 0, & 0 \leqq t < \tau_3\,(x_0)\,, \\ K - x_4^*, & t \geqq \tau_3\,(x_0)\,, \end{cases} & 0 < x_0 < x_4^*. \end{cases} \tag{C.17}$$

□

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